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The reducibility of the anisotropic Hele-Shaw problem to the isotropic case ☆

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Abstract

It is established that the unilateral Hele-Shaw problem for flows in a channel when there is bulk anisotropy and Saffman–Taylor boundary conditions on the free boundary can be reduced to the isotropic case using a linear non-orthogonal coordinate transformation. Correspondingly, any exact solution of the Hele-Shaw problem for an isotropic medium generates a set of solutions for an anisotropic medium for arbitrary orientation of the principal axes of the permeability tensor with respect to the direction of the channel axis.

References in the literature on anisotropic Hele-Shaw flows usually related to the case of a regular array of grooves etched on one of the glass plates forming a cell. The motion of the liquid in this case is described by the same Laplace equation, i.e. there is actually no bulk anisotropy of the medium, and only the form of the capillary-type condition at the interface changes, so that there is a certain analogy between the fingering processes in such Hele-Shaw flows and dendrite growth processes.[1] Nevertheless, in the production of certain composite materials, for example, in the vacuum infusion of a resin into closed forms, capillary effects at the interface are unimportant, but in return the bulk anisotropy of the medium is extremely important.[2]

For the case of an isotropic medium, many accurate stationary and non-stationary solutions of the Hele-Shaw problem with dynamic Saffman–Taylor boundary conditions on the free boundary are known (see, for example, Refs. 3–5), including of fairly general form.[6] At the same time, for the case of an anisotropic medium there is obviously a unique exact solution of the problem with a free boundary of the Hele-Shaw problem type – the solution of the stationary problem of the influx of ground waters to a drain in a lock in an anisotropic ground.[7] The construction of this solution is based on the use of the well-known linear non-orthogonal coordinate transformation.[8,9] The purpose of the present paper is to investigate what the use of this transformation contributes to the Hele-Shaw problem with a free boundary in the general non-stationary formulation.

1. Formulation of the problem

As is well known,[7,9] a Hele-Shaw cell models two-dimensional flows of a viscous incompressible fluid in porous media, provided they follow Darcy's law. For anisotropic uniform media the permeability is a tensor constant **K**. We will

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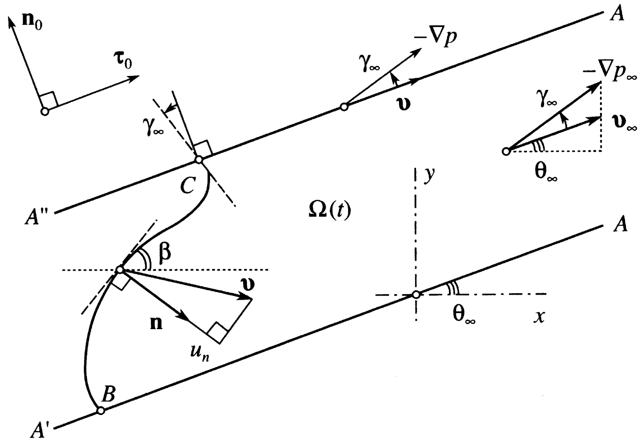

Fig. 1.

consider a Hele-Shaw cell of the channel type (Fig. 1). We will connect the origin of a Cartesian system of coordinates x, y with some fixed point of the channel wall, and we will direct axes along the principal axes of anisotropy of the permeability tensor. Then

$$\mathbf{K} = \begin{Vmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{Vmatrix}$$

The flow rate $\mathbf{v}$ satisfies the incompressibility condition $\nabla\mathbf{v} = 0$, and its components are related to the pressure by Darcy's law

$$\upsilon_x = -k_{xx}\mu^{-1}\partial p/\partial x, \quad \upsilon_y = -k_{yy}\mu^{-1}\partial p/\partial y \tag{1.1}$$

Here μ is the viscosity and $p(x, y, t)$ is the pressure of the fluid at the point x, y at the instant of time t.

For flows in Hele-Shaw cells of the channel type it is natural to choose the width of the channel of the characteristic length l^*, and the fluid velocity at infinity as the characteristic velocity v^*, in which case $t^* = l^*/v^*$ is the characteristic time. An analysis of the dimensions of Eq. (1.1) suggests the choice of the characteristic pressure: $p^* = l^* v^* \mu / k_{xx}$, and also gives the dimensionless complex $k = \sqrt{k_{yy}/k_{xx}}$, characterizing the anisotropy of the medium.

Referring the dimensional variables x, y, t, p and $\mathbf{v}$ to the characteristic quantities we change to dimensionless variables, keeping the same notation for them as for the dimensional variables. Thus, Eq. (1.1) give the components of the dimensionless velocity $\mathbf{v}$

$$\upsilon_x = -\partial p/\partial x, \quad \upsilon_y = -k^2\partial p/\partial y \tag{1.2}$$

and we can write the dimensionless formulation of the problem.

The fluid occupies the region $\Omega(t)$ with boundary $\partial\Omega(t) = AB \cup BC \cup CA$. The part $BC \equiv \Gamma(t)$ is the interface, its configuration is unknown and is to be determined in the course of solving the problem. The parts $AB \subset AA'$, $CA \subset AA''$ are parts of the channel walls, the direction of which, generally speaking, does not coincide with any of the directions of the principal axes of anisotropy x, y. We will denote the angle between the direction of the x axis and the direction of the walls by θ_∞ and we will assume that they belong to the first quadrant $\theta_\infty \in [0, \pi/2)$ (this can always be achieved by an appropriate choice of the coordinate axes). We have

$$AA'\colon\ y = x\,\mathrm{tg}\,\theta_\infty; \quad AA''\colon\ y = x\,\mathrm{tg}\,\theta_\infty + 1/\cos\theta_\infty \tag{1.3}$$

We will agree that $\tau_0 = (\cos\theta_\infty, \sin\theta_\infty)$ is the unit vector of the direction of the channel axis along the path of the fluid motion, while $\mathbf{n}_0 = (-\sin\theta_\infty, \cos\theta_\infty)$ is the unit vector of the normal to this axis.

The incompressibility condition, taking expressions (1.2) into account, gives the equation for the function $p(x, y, t)$

$$(x, y) \in \Omega(t): \partial^2 p/\partial x^2 + k^2 \partial^2 p/\partial y^2 = 0 \tag{1.4}$$

The impermeability condition $\mathbf{v} \cdot \mathbf{n}_0 = 0$ is satisfied on the channel walls; it can be written, using expressions (1.2), in the form

$$AB \cup CA: \sin\theta_\infty \partial p/\partial x - k^2 \cos\theta_\infty \partial p/\partial y = 0 \tag{1.5}$$

At infinity, i.e. at the point A, the condition $\mathbf{v}_\infty = \tau_0$ must be satisfied or, like the previous case,

$$(x, y) \to A: (\partial p/\partial x, \partial p/\partial y) \to -(\cos\theta_\infty, k^{-2}\sin\theta_\infty) \tag{1.6}$$

Two conditions are satisfied on the free boundary $\Gamma(t)$.[3] The first is a dynamic condition (the Saffman–Taylor condition)

$$\Gamma(t): p = 0 \tag{1.7}$$

The second is a kinematic condition, which most often of all is written in the form

$$\Gamma(t): u_n = \upsilon_x \sin\beta - \upsilon_y \cos\beta$$

Here u_n is the normal velocity of displacement of the boundary, β is the angle of inclination to the horizontal of the tangent to the boundary $\Gamma(t)$ (the tangent is shown by the dashed line in Fig. 1), while on the right-hand side of the relation there is the projection of the fluid velocity $\mathbf{v}$ onto the normal to the boundary $n = (\sin\beta, -\cos\beta)$. However, the quantities β and u_n are difficult to formalize in terms of the function $p(x, y, t)$, and hence in this case it is more convenient to use another form of writing this condition[4]

$$\Gamma(t): \partial p/\partial t + \upsilon_x \partial p/\partial x + \upsilon_y \partial p/\partial y = 0$$

which expresses the fact that the total derivative with respect to time of the pressure function on the free boundary is equal to zero. Note that, in the theory of Stefan's problem, which is related to the Hele-Shaw problem (the temperature occurs in it instead of the pressure), this is one of the forms of writing Stefan's condition.[10,11] Expressing it using the same formulae (1.2) in terms of the function $p(x, y, t)$, we obtain the kinematic condition on the free boundary

$$\Gamma(t): \partial p/\partial t = (\partial p/\partial x)^2 + k^2(\partial p/\partial y)^2 \tag{1.8}$$

Among other things, this form of notation emphasises the non-linear form of the free-boundary problem.

To complete the formulation of the problem it is necessary to specify the configuration of the region $\Omega(t)$ at the initial instant of time $t = 0$

$$\Omega(t)|_{t=0} = \Omega_0 \tag{1.9}$$

The system of Eqs. (1.3)–(1.9) is a dimensionless formulation of the Hele-Shaw boundary-value problem with a Saffman–Taylor type dynamic condition on the non-stationary free boundary for flow in a channel when there is both anisotropy, formalized in terms of a single unknown function $p(x, y, t)$. The special case $k \equiv 1$ corresponds to a uniform isotropic medium.

An analysis of the above problem enables us to establish some qualitative properties of its possible solutions. Knowing the components of the vectors $\mathbf{v}_\infty$ and $-\nabla p_\infty$ we can obtain the angle between them (see Fig. 1)

$$\gamma_\infty = \operatorname{arctg}(k^{-2}\operatorname{tg}\theta_\infty) - \theta_\infty; \quad \gamma_\infty \in [-\pi/2; \pi/2] \tag{1.10}$$

An estimate of the range of variation of the angle was made taking into account the fact that $\theta_\infty \in [0, \pi/2)$.

We will now present the simplest case of a uniform (piston) displacement with a fluid velocity everywhere equal to $\mathbf{v} = \tau_0$. Then, over the whole flow region and, in particular, on the channel walls the angle between the vectors $-\nabla p$ and $\mathbf{v}$ will be equal to γ_∞. Further, we will take into account the fact that the free boundary corresponds to the condition $p = 0$ and, consequently, is orthogonal to the vector ∇p at each point of it. Then, at the ends B and C the angle between the free boundary and the normal n_0 to the walls will also be equal to γ_∞. Moreover, in view of the uniformity of the flow, these angles are independent of the value of the fluid velocity.

Transferring to the case of essentially two-dimensional flow, we note that at any internal point of the region $\Omega(t)$ the velocity vector $\mathbf{v}$, generally speaking, differs both in value and direction from the vector $\mathbf{v}_\infty$. At the same time, on the walls the vectors $\mathbf{v}$ and $\mathbf{v}_\infty$ are always codirected, and locally close to the walls the flow is always uniform (the value of the velocity, naturally, is necessarily equal to unity), and hence the conclusion that on the channel walls the angle between the vectors $-\nabla p$ and $\mathbf{v}$ is equal to γ_∞ holds, which could also be formally proved using expressions (1.6) and (1.10). The conclusion that, at the ends B and C of the free boundary $\Gamma(t)$, the angle formed by the free boundary and the normal $\mathbf{n}_0$ to the walls is also equal to γ_∞ (see Fig. 1) similarly holds.

2. Transfer to new coordinates

Consider the transformation of the coordinates x, y, t into coordinates X, Y, T

$$x = C_1 k^{-1} X, \quad y = C_1 Y, \quad t = C_2 T \tag{2.1}$$

where C_1 and C_2 are so-far undefined positive constants. This is essentially a simple generalization of a linear non-orthogonal transformation of the coordinates,[7–9] which converts Eq. (1.4) into Laplace equation. In particular, it converts representation (1.3) of the channel walls $AA' \cup AA''$ into the representation

$$\begin{gathered} AA'\colon Y = X\operatorname{tg}\Theta_\infty; \quad AA''\colon Y = X\operatorname{tg}\Theta_\infty + 1/(C_1\cos\theta_\infty) \\ \Theta_\infty = \operatorname{arctg}(k^{-1}\operatorname{tg}\theta_\infty); \quad \Theta_\infty \in [0, \pi/2) \end{gathered} \tag{2.2}$$

i.e. the rectilinear channel walls in the new coordinates X, Y convert into rectilinear walls inclined to the horizontal at an angle Θ_∞.

Further, we will define a new function $P(X, Y, T)$

$$p(x, y, t) = C_3^{-1} P(X, Y, T) \tag{2.3}$$

where C_3 is a so-far undefined positive constant. We will analyse how the Hele-Shaw problem (1.3)–(1.9), formulated in Section 1, is altered on changing to the new coordinates X, Y, T and the new function $P(X, Y, T)$.

In the X, Y plane the region $\Omega(T)$ will be limited by the channel walls (2.2) and the free boundary $BC \equiv \Gamma(T)$. The function $P(X, Y, T)$ will satisfy the equation

$$\Omega(T)\colon \partial^2 P/\partial X^2 + \partial^2 P/\partial Y^2 = 0 \tag{2.4}$$

the boundary conditions on the channel walls

$$AB \cup CA\colon \sin\Theta_\infty \partial P/\partial X - \cos\Theta_\infty \partial P/\partial Y = 0 \tag{2.5}$$

the condition at infinity

$$(X, Y) \to A\colon (\partial P/\partial X, \partial P/\partial Y) \to -(C_3 C_1 k^{-1}\cos\theta_\infty, C_3 C_1 k^{-2}\sin\theta_\infty) \tag{2.6}$$

the two conditions on the free boundary

$$\Gamma(T)\colon P = 0, \quad \partial P/\partial T = [C_2 k^2/(C_3 C_1^2)][(\partial P/\partial X)^2 + (\partial P/\partial Y)^2] \tag{2.7}$$

and the initial condition

$$\Omega(0) = \Omega_0 \tag{2.8}$$

In the formulation of the problem in terms of the function $P(X, Y, T)$, Eq. (2.4), boundary condition (2.5), the first condition of (2.7) and also the initial condition (2.8) correspond to the isotropic case of the formulation of the Hele-Shaw problem in Section 1. It can be shown that, by choosing the constants C_1, C_2 and C_3 in a special way, the remaining boundary conditions, namely, condition (2.6), the second condition of (2.7) and also representation (2.2) of the channel walls, can also be made to correspond to the isotropic case.

In fact, in formula (2.2) the constant C_1 obviously corresponds to the width of the channel in the new system of coordinates (X, Y). One can obtain unit width of the channel and, of course, a complete analogy between representations (2.2) and (1.3) of the channel walls, by choosing as C_1 the quantity

$$C_1 = k(\sin^2\theta_\infty + k^2\cos^2\theta_\infty)^{-1/2} \tag{2.9}$$

Now, using relation (2.9) successively and the notation Θ_∞ in (2.2), it can be shown that the following expression holds

$$C_1\sin\theta_\infty = k\sin\Theta_\infty$$

Substituting it into boundary condition (2.6), we obtain the simpler form of the condition

$$(X, Y) \to A: (\partial P/\partial X, \partial P/\partial Y) \to -C_3k^{-1}(\cos\Theta_\infty, \sin\Theta_\infty)$$

It is obvious that complete correspondence between the conditions at infinity and the isotropic case can be achieved by choosing

$$C_3 = k \tag{2.10}$$

Finally, we can obtain exact correspondence between the second boundary condition on the free boundary (2.7) and the isotropic case by choosing C_2 in the form

$$C_2 = k(\sin^2\theta_\infty + k^2\cos^2\theta_\infty)^{-1} \tag{2.11}$$

3. Discussion of the results and an example

As a result of the transformation of coordinates (2.1) and function (2.3) with the appropriate choice (2.9)–(2.11) of the constants C_1, C_2 and C_3, the Hele-Shaw problem (1.3)–(1.9) for the fluid flow in a channel when there is anisotropy can be reduced to the analogous problem for an isotropic medium. Consequently, any exact solution of the Hele-Shaw problem for flows in an isotropic cell generates a set of solutions for flows in an anisotropic cell for any orientation of the principal axes of the permeability tensor with respect to the direction of the channel axis. The result can be extended in an obvious way to stationary solutions, and also to Hele-Shaw flow in cells with a circular geometry (the problem of an "inflating bubble"), since in this case there are no impenetrable channel walls and one can simply dispense with the need to keep track of the boundary conditions on them.

As an example we will take the exact solution describing the formation of a single symmetrical finger in a Hele-Shaw channel.[4] We will confine ourselves to the simplest case when, over a long period of time, the well-known Saffman–Taylor finger is formed with a width equal to the half-width of the channel. The complex flow potential $W = -P + i\Psi$ as a function of the point $Z = X + iY$ of the complex physical plane at the instant of time T is implicitly defined by the equation[4]

$$\begin{aligned} X + iY &= F(T, P, \Psi) \\ F(T, P, \Psi) &= \\ &= e^{i\Theta_\infty}\left\{T + \frac{1}{4\pi}\ln(1+\alpha e^{4\pi T}) - P + i\Psi + \frac{1}{2\pi}\ln\frac{1+(1+\alpha^{-1}e^{-4\pi T})^{-1/2}e^{2\pi(P-i\Psi)}}{1+(1+\alpha^{-1})^{-1/2}}\right\} + \mathbb{C}_0 \end{aligned} \tag{3.1}$$

with the condition $-1/2 \le \Psi \le 1/2$, $-\infty < P \le 0$. Here α is an arbitrary positive constant, $\mathbb{C}_0$ is an unimportant complex constant (its choice fixes the origin of coordinates), and Θ_∞ is the angle between the channel axis and the X axis, the value of which will be determined later depending on the specified parameters of the anisotropic case. In particular, Eq. (3.1) also implicitly defines the pressure function

$$P(X, Y) = -\mathrm{Re}\,W$$

If we put $P = 0$, Eq. (3.1) becomes a parametric equation of the interface

$$\Gamma(t): X = X(\Psi), \quad Y = Y(\Psi)$$

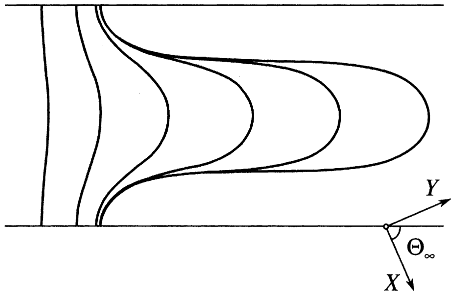

Fig. 2.

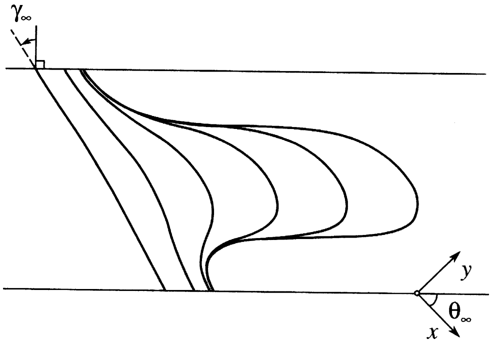

Fig. 3.

Correspondingly, we can follow the evolution of the interface. The choice of a definite value of α obviously fixes the initial configuration of the free boundary $\Gamma(0)$. We choose $\alpha = 0.01$. Then the first 5 steps of $\Delta T = 0.2$ in time T give the evolution pattern of the interface, shown in Fig. 2.

Further, we take a Hele-Shaw channel having bulk anisotropy. To fix our ideas, we will specify the angle θ_∞ between the channel axis and one of the principal anisotropy axes of the permeability tensor (it is already the x axis): $\theta_\infty = \pi/4$. Suppose $k_{yy} = k_{xx}/4$, i.e. the anisotropy parameter $k = 1/2$. Then, by the formulae in Section 2, we obtain

$$\gamma_\infty = \operatorname{arctg} 0.6, \quad \Theta_\infty = \operatorname{arctg} 2, \quad C_1 = \sqrt{0.4}, \quad C_2 = 0.8, \quad C_3 = 0.5$$

whence we obtain the transition formulae

$$x = 2\sqrt{0.4}X, \quad y = \sqrt{0.4}Y, \quad t = 0.8T, \quad p(x, y, t) = 2P(X, Y, T)$$

As a result, the exact solution for an isotropic Hele-Shaw channel (3.1) produces the exact solution for an anisotropic channel. In this case the evolution of the interface for an anisotropic Hele-Shaw channel can be represented by the parametric equation

$$x(\Psi)/2 + iy(\Psi) = \sqrt{0.4}F(t/0.8, 0, \Psi)$$

Hence, the first 5 steps of $\Delta t = 0.16$ in time t give the evolution pattern of the interface shown in Fig. 3.

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